Completing the Square — Prepared by Nikolas Agisilaou

What Is Completing the Square?

Rewrite ax^2+bx+c as $a(x+p)^2+q$.

- **Core trick:** constant $= \left(\frac{\text{coeff of } x}{2}\right)^2$
- Reveals solutions, vertex, min/max.
- Works on *any* quadratic.
- $y=a(x-h)^2+k$: vertex (h,k); $a>0$: minimum $a<0$: maximum

Perfect Square Identities — The Foundation

$(x+a)^2 = x^2+2ax+a^2$ $\quad (x-a)^2 = x^2-2ax+a^2$

$(x+7)^2 = x^2+14x+\mathbf{49}$ $\quad \left(\frac{14}{2}\right)^2 = 49$ $\quad (x-6)^2 = x^2-12x+\mathbf{36}$

$\left(x+\frac{3}{2}\right)^2 = x^2+3x+\frac{\mathbf{9}}{4}$ $\quad \left(\frac{3}{2}\right)^2 = \frac{9}{4}$ $\quad$ *Bold constant is always* $\left(\frac{b}{2}\right)^2$

The Two Methods: Step-by-Step

Case 1 — $a=1$: x^2+bx+c

1. Half the x-coefficient: $p=\dfrac{b}{2}$
2. $(x+p)^2$ expands to x^2+bx+p^2
3. Subtract p^2, keep c: $(x+p)^2-p^2+c$
4. Simplify: $q=c-p^2$

Formula: $x^2+bx+c=\left(x+\dfrac{b}{2}\right)^2-\left(\dfrac{b}{2}\right)^2+c$

Case 2 — $a\neq 1$: ax^2+bx+c

1. Factor a from x-terms *only*: $a\left(x^2+\frac{b}{a}x\right)+c$
2. Complete the square inside the bracket
3. Multiply a back through the bracket
4. Simplify the constants

Formula: $ax^2+bx+c=a\left(x+\dfrac{b}{2a}\right)^2-\dfrac{b^2}{4a}+c$

Worked Examples at a Glance

Expression	Key step	Result	Expression	Key step	Result
$x^2+10x+3$	$p=5,\ p^2=25$	$(x+5)^2-22$	$2x^2+16x+3$	$2(x^2+8x);\ 2\times 16=32$	$2(x+4)^2-29$
$x^2-12x+11$	$p=-6,\ p^2=36$	$(x-6)^2-25$	$4x^2-24x+11$	$4(x^2-6x);\ 4\times 9=36$	$4(x-3)^2-25$
x^2+9x+4	$p=\frac{9}{2},\ p^2=\frac{81}{4}$	$\left(x+\frac{9}{2}\right)^2-\frac{65}{4}$	$-x^2+8x+3$	$-(x^2-8x);\ -(-16)=+16$	$-(x-4)^2+19$

Solving Equations by Completing the Square

Steps: complete the square → rearrange → take $\pm\sqrt{\ }$

$(x+p)^2=k \implies x=-p\pm\sqrt{k}$

$x^2+10x+3=0$ $\quad (x+5)^2=22 \Rightarrow x=-5\pm\sqrt{22}$

$2x^2-12x+7=0$ $\quad 2(x-3)^2=11 \Rightarrow x=3\pm\dfrac{\sqrt{22}}{2}$

$x^2+6x+15=0$ $\quad (x+3)^2=-6 \Rightarrow$ *no real solutions*

Number of Solutions & Discriminant Link

After completing the square: $(x+p)^2=k$

k	Solutions	Graph
$k>0$	Two distinct real roots	Cuts x-axis
$k=0$	One repeated root	Touches x-axis
$k<0$	No real roots	Misses x-axis

Same rules with $\Delta=b^2-4ac$: $\Delta>0,=0,<0$.

Deriving the Quadratic Formula — Completing the Square on ax^2+bx+c

Step 1: Divide by a: $x^2+\frac{b}{a}x=-\frac{c}{a}$

Step 2: Add $\left(\frac{b}{2a}\right)^2$ to both sides

Step 3: $\left(x+\frac{b}{2a}\right)^2=\dfrac{b^2-4ac}{4a^2}$

Step 4: $x+\frac{b}{2a}=\pm\dfrac{\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Common Traps & Pitfalls

1. **Subtract Trap:** $(x+5)^2+3\neq x^2+10x+3$. Always subtract p^2.
2. **Sign Trap:** Half of -12 is $-6\Rightarrow(x-6)^2$, not $(x+6)^2$.
3. **Factor Trap:** Factor a from x-terms *only* — not the constant.
4. **Negative a Trap:** $-[(x-4)^2-16]=-(x-4)^2+16$.
5. **$\pm$ Trap:** Never drop the $\pm$ when taking a square root to solve.

Quick Exam Shortcuts

"Form $(x+a)^2+b$" → Complete the square

"Surd-form answers" → Complete the square

"No real solutions" → Show $(x+p)^2=k<0$

"Derive quad. formula" → CTS on $ax^2+bx+c=0$

Always: expand back to verify the original.

Odd-b values of p^2: $\frac{1}{4},\ \frac{9}{4},\ \frac{25}{4},\ \frac{49}{4},\ \frac{81}{4}\ldots$

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