Rational & Irrational Numbers

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Rational Numbers (The Exact Fractions)

A number is **rational** if it can be written as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.

- **Integers:** $8 = \frac{8}{1}$, $-5 = \frac{-5}{1}$, $0 = \frac{0}{1}$.
- **Terminating decimals:** $0.85 = \frac{17}{20}$.
- **Recurring decimals:** $0.\overline{6} = \frac{2}{3}$.

Irrational Numbers (The Infinite Non-Repeaters)

A number is **irrational** if it **cannot** be written as $\frac{p}{q}$ for any integers p, q.

- **Decimal:** Goes on forever with **no** repeating block.
- **Key examples:** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, $\sqrt{11}$, π.
- **Rule:** $\sqrt{n}$ is irrational when n is **not** a perfect square.

The Number System & Quick Identification

$\mathbb{N}$ Natural: $0, 1, 2, 3, \ldots$

$\mathbb{Z}$ Integers: $\ldots, -2, -1, 0, 1, 2, \ldots$

$\mathbb{Q}$ Rational: any $\frac{p}{q}$ $(q \neq 0)$

$\mathbb{R}$ Real: rationals + irrationals

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

If a number is...	Then it is
An integer or fraction	Rational
A terminating decimal	Rational
A recurring decimal	Rational
$\sqrt{n}$, n not a perfect square	Irrational
π or a multiple of π	Irrational

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Converting Recurring Decimals to Fractions

1. Let $x =$ the recurring decimal
2. Multiply by 10^k where $k =$ number of repeating digits
3. Subtract the equations to cancel the recurring part
4. Solve for x and simplify the fraction

Mixed case (e.g. $0.3\overline{27}$): multiply by 10 *then* 1000, subtract $1000x - 10x$.

Worked Example

$$x = 0.\overline{54} = 0.5454\ldots$$
$$100x = 54.5454\ldots$$
$$100x - x = 54$$
$$99x = 54$$
$$x = \frac{54}{99} = \frac{6}{11}$$

Operations: Closure Properties

Rational numbers are **closed** under $+, -, \times, \div$ (except $\div 0$): rational $\circ$ rational = rational.

- **Rational + Irrational = Irrational**: e.g. $7 + \sqrt{5}$ is irrational.
- **Rational $\times$ Irrational = Irrational** (if rational $\neq 0$): e.g. $6\sqrt{3}$ is irrational.
- **Irrational $\pm$ Irrational**: *either!* $\sqrt{7} + (-\sqrt{7}) = 0$ (rational) vs. $\sqrt{3} + \sqrt{11}$ (irrational).

- **Irrational** $\times$ **Irrational**: *either!* $\sqrt{5} \times \sqrt{5} = 5$ (rational) vs. $\sqrt{3} \times \sqrt{7} = \sqrt{21}$ (irrational).

Common Traps & Pitfalls

1. **The π Trap:** $\frac{22}{7}$ and $\frac{355}{113}$ are **rational** approximations. They are **not** equal to π (which is irrational).
2. **The Root Trap:** Not all roots are irrational! $\sqrt{81} = 9$ and $\sqrt[3]{27} = 3$ are rational.
3. **The "Long Decimal" Trap:** $0.\overline{153846}$ looks complex but $= \frac{2}{13}$ — rational, because it recurs.
4. **The Irrational $\times$ Irrational Trap:** Result isn't always irrational: $\sqrt{12} \times \sqrt{3} = \sqrt{36} = 6$.

Quick Exam Shortcuts

Is it rational? Try to write it as $\frac{p}{q}$. If decimal never terminates *and* never repeats $\rightarrow$ irrational.
Power-of-10 rule: 1 repeating digit $\rightarrow \times 10$; 2 digits $\rightarrow \times 100$; 3 digits $\rightarrow \times 1000$.
Perfect squares to know: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225$.
Key irrationals: $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$, $\sqrt{5} \approx 2.236$, $\pi \approx 3.14159$.

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